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QUASI-PLANAR GRAPHS HAVE A LINEAR NUMBER OF EDGES

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A graph is called *quasi-planar* if it can be drawn in the plane so that no three of its edges are pairwise crossing. It is shown that the maximum number of edges of a quasi-planar graph with n vertices is $O(n)$.

1. Introduction

We say that an undirected graph $G(V,E)$ without loops or parallel edges is *drawn* in the plane if each vertex $v \in V$ is represented by a distinct point and each edge $e=(u,v) \in E$ is represented by a Jordan arc connecting the points representing u and v. Throughout this paper, we assume that any two arcs of a drawing have at most one point in common, which is either a common endpoint or a common

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interior point where the two arcs cross each other. We do not make any notational distinction between vertices of G and the corresponding points in the plane, or between edges of G and the corresponding Jordan arcs.

A graph that can be drawn in the plane without crossing edges is planar. We call a graph *quasi-planar* if it can be drawn in the plane with no three pairwise crossing edges. The aim of this paper is to establish an $O(n)$ upper bound on the number of edges in any quasi-planar graph with n vertices. This improves an earlier result of Pach et al. [7], who had shown that a quasi-planar graph with n vertices has $O(n\log^2 n)$ edges. This is a special case of a more general result, in which one assumes that G can be drawn in the plane with no k pairwise crossing edges, for some constant $k \geq 3$. It is shown in [7] that in this case the number of edges of G is $O(n\log^{2k-4} n)$. Using our main result, we also improve these bounds to $O(n\log^{2k-6} n)$, as shown in Section 3. A linear upper bound was proved by Capoyleas and Pach [1] for a special case of this problem. Many other related problems and results can be found in the book by Pach and Agarwal [6].

Theorem 1.1. *If $G(V,E)$ is a quasi-planar graph, then $|E| = O(|V|)$.*

We prove this theorem in Section 2, and in Section 3 we consider some related problems and generalizations.

2. Proof of Theorem 1.1

To simplify our presentation, we prove the theorem only in the special case when G has a straight-line drawing with no three pairwise crossing edges (straight line segments). Remarkably, the proof for the general case requires only minor modification.

The set of edges $E = E(G)$ defines a cell complex in the plane, whose 0-, 1-, and 2-dimensional cells will be called *nodes*, *segments*, and *faces*, respectively. This cell complex is known as the *arrangement* of E and is denoted by $\mathcal{A}(E)$. More specifically, the nodes of $\mathcal{A}(E)$ are the endpoints and crossings of graph edges, a segment of $\mathcal{A}(E)$ is a portion of a graph edge between two consecutive nodes, and a face of $\mathcal{A}(E)$ is a connected component of the complement of the union of E. (To avoid ambiguity, we hereafter refer to vertices and edges of G, and to the corresponding points and line segments in the plane, as "graph vertices" and "graph edges," respectively.) Let X be the set of crossings of graph edges, $N = V \cup X$ the set of all nodes of $\mathcal{A}(E)$, S the set of its segments, and F the set of its faces. For a face $f \in F$, the *complexity* $|f|$ of f is the number of segments of S on the boundary ∂f of f. As customary, if both sides of a segment are incident to the interior of f, then it contributes 2 to $|f|$. Let $t(E) = |\{f \in F : |f| = 3\}|$ be the number of triangular faces in F.

Lemma 2.1. *Let $G(V,E)$ be a graph drawn in the plane. Then the total complexity of all non-quadrilateral faces of the arrangement $\mathcal{A}(E)$ is at most $8t(E)+20|V|$.*

Proof. It is sufficient to prove the lemma with the assumption that the planar graph (N,S) is connected and $|S|>1$.

Recall the following familiar facts:

$$\sum_{f\in F} |f| = 2|S|,$$

$$2|S| = \sum_{v\in N} \deg(v) = \sum_{v\in V} \deg(v) + \sum_{v\in X} \deg(v) \geq 2|E| + 4|X|,$$

and

$$|V| + |X| + |F| = |N| + |F| = |S| + 2.$$

The first two lines just express two different ways of counting the edges of the planar graph (N,S), as the sum of face complexities and of vertex degrees, respectively. The third line is Euler's relation. These easily yield

$$\sum_{f\in F} |f| \leq 4|V| + 4|F| - 2|E| - 8, \quad \text{hence} \quad \sum_{f\in F} (|f| - 4) \leq 4|V|.$$

Finally, since $|f|>4$ implies $|f|\leq 5(|f|-4)$, we have

$$\sum_{f\in F, |f|\neq 4} |f| \leq 3t(E)+5 \sum_{f\in F, |f|\geq 4} (|f|-4) = 8t(E)+5\sum_{f\in F} (|f|-4) \leq 8t(E)+20|V|.$$

∎

Lemma 2.2. *Let $G(V,E)$ be a quasi-planar graph drawn in the plane. Then the overall complexity of all faces f of $\mathcal{A}(E)$, such that f is either a non-quadrilateral face or a quadrilateral face incident to at least one vertex of G, is $O(|V|+|E|)$.*

Proof. We first claim that $t(E)=O(|E|)$. Indeed, each triangular face f of $\mathcal{A}(E)$ must be incident to a vertex of G, for otherwise there would be three pairwise crossing edges in G. It is easy to check that the number of faces of $\mathcal{A}(E)$ incident to graph vertices is at most $2|E|$, so our claim follows. In addition, this implies that the overall complexity of all quadrilateral faces of $\mathcal{A}(E)$ incident to a graph vertex in V is also $O(|E|)$. The lemma is now an immediate consequence of Lemma 2.1. ∎

Let $G(V,E)$ be a quasi-planar graph drawn in the plane with $n=|V|$ vertices. Returning to the proof of Theorem 1.1, we may assume without loss of generality that G is connected, because it suffices to establish a linear bound on the number of graph edges in each connected component of G. Let $G_0=(V,E_0)$ be a spanning tree of G, so $|E_0|=n-1$. Let $E^*=E\setminus E_0$. Note that each face of the arrangement $\mathcal{A}(E_0)$ is simply connected, for otherwise the union of nodes and segments of $\mathcal{A}(E_0)$ would

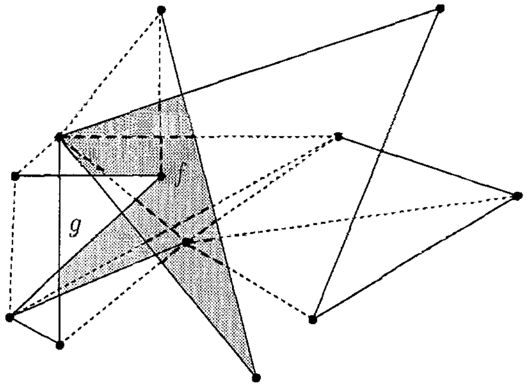

Fig. 1. The arrangement $\mathcal{A}(E_0)$: the edges of E_0 are drawn in solid; the face g is a crossing quadrilateral; the edges of E^* are drawn as dotted, and their portions within the face f, forming $X(f)$, are dashed

not be connected, contradicting the connectedness of G_0. Moreover, by Lemma 2.2, the complexity of all those faces of $\mathcal{A}(E_0)$ which are either non-quadrilaterals or quadrilaterals incident to a point in V is $O(n)$. We refer to the remaining faces of $\mathcal{A}(E_0)$ as *crossing quadrilaterals.* See Figure 1 for an illustration.

In the sequel, we use the following notion. A graph is called an *overlap graph* if its vertices can be represented by intervals on a line so that two vertices are adjacent if and only if the corresponding intervals overlap but neither of them contains the other [2]. Gyárfás [3] (see also [4]) has shown that every triangle-free overlap graph can be colored by a constant number, c, of colors, and Kostochka [5] proved that this is true with $c=5$.

For each graph edge $e \in E^*$, let $\Xi(e)$ denote the set of segments of $\mathcal{A}(E_0 \cup \{e\})$ that are contained in e. In other words, it is the set of segments into which e is cut by the graph edges of E_0. By construction, each segment $s \in \Xi(e)$ is fully contained in a face $f \in \mathcal{A}(E_0)$ and its two endpoints lie on ∂f. For each face f of $\mathcal{A}(E_0)$, let $X(f)$ denote the set of all segments in $\bigcup_{e \in E^*} \Xi(e)$ that are contained in f, and let $H(f)$ denote the quasi-planar graph whose set of edges is $X(f)$. Since f is simply connected, any two segments in $X(f)$ cross each other if and only if their endpoints alternate along the boundary of f. By cutting the boundary of f so that it becomes an (image of an open) interval and associating with each segment in $X(f)$ the connected subinterval along the boundary of f between its endpoints, we obtain a collection of intervals with the property that two elements of $X(f)$ cross if and only if the corresponding intervals overlap and neither is contained in the other. This defines a triangle-free overlap graph on the vertex set $X(f)$. Therefore, the segments of $X(f)$ can be colored by at most five colors, so that no two segments with the same color cross each other. (Note that, for a graph edge $e \in E^*$, several segments in $\Xi(e)$ may be contained in the same face f and thus belong to the same $X(f)$. These segments may be colored by different colors.)

Let f be a face of $\mathcal{A}(E_0)$ other than a crossing quadrilateral, and let $H_1(f), \ldots, H_5(f)$ be the monochromatic subgraphs of $H(f)$ obtained by the above coloring. Fix one of these subgraphs, say $H_1(f)$, and re-interpret it as a graph whose vertices are the (relative interiors of the) edges of ∂f together with the elements of V on ∂f, and whose edges are the segments of $H_1(f)$. The resulting graph, $H_1^*(f)$, is clearly planar. We call a face of $H_1^*(f)$ a *digon* if it is bounded by exactly two edges of $H_1^*(f)$, and we call an edge of $H_1^*(f)$ *shielded* if both of the faces incident to it are digons. The remaining edges of $H_1^*(f)$ are called *exposed*. See Figure 2. Observe that, by Euler's formula, there are at most $O(n_f)$ exposed edges in $H_1^*(f)$, where n_f is the number of vertices of $H_1^*(f)$, which is at most $2|f|$.

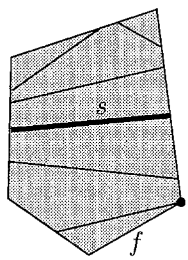

Fig. 2. A monochromatic subgraph $H_1(f)$ of the graph $H(f)$. The segment s is a shielded edge of $H_1^*(f)$; all other edges of $H_1^*(f)$ are exposed

We repeat this analysis for each of the other subgraphs $H_2(f), \ldots, H_5(f)$, and for all faces f of $\mathcal{A}(E_0)$ other than crossing quadrilaterals. It follows that the number of graph edges $e \in E^*$ containing at least one exposed segment (in the graph $H_i^*(f)$ containing it) is $O(\sum_f |f|)$, where the sum extends over all such faces f. By Lemma 2.2, this sum is $O(n)$.

It thus remains to bound the number of graph edges in E^* with no exposed subsegment; we call these edges *shielded*, borrowing the terminology used above. If e is a shielded graph edge, then, for each $s \in \Xi(e)$, either s lies in a crossing quadrilateral face of $\mathcal{A}(E_0)$, or else s is shielded (in the corresponding subgraph). Note that no graph edge $e \in E^*$ can consist solely of segments passing through crossing quadrilaterals, as the first and last segments necessarily meet faces of $\mathcal{A}(E_0)$ that have at least one graph vertex on their boundary, namely an endpoint of e.

Lemma 2.3. *There are no shielded graph edges.*

Proof. Suppose that $e \in E^*$ is shielded. Let a and b be the endpoints of e, and let $s_1, \ldots, s_k$ denote the segments in $\Xi(e)$, appearing along e in their order from a to b. We claim that there exists a graph edge $e^+ \in E^*$ that satisfies the following properties:

(1) e^+ is a graph edge of E^* emanating from a next to e, and

(2) for each segment $s_i \in \Xi(e)$, there is a corresponding segment $s_i^+ \in \Xi(e^+)$, such that s_i and s_i^+ connect the same pair of segments of $\mathscr{A}(E_0)$, for $i=2, \ldots, k-1$, s_1 and s_1^+ connect a to the same segment of $\mathscr{A}(E_0)$, and s_k and s_k^+ connect the same segment of $\mathscr{A}(E_0)$ to b.

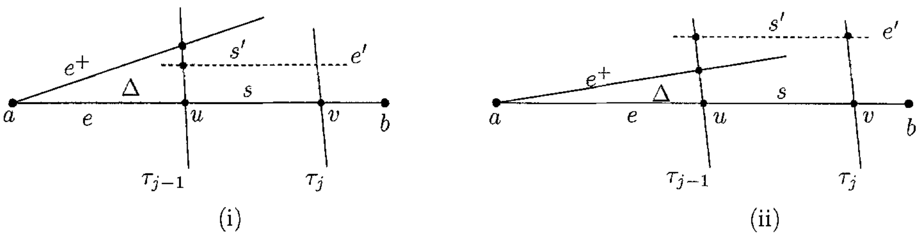

Illustration to the proof of Lemma 2.3: (i) e' crosses τ_{j-1} between e and e^+; (ii) e^+ crosses τ_{j-1} between e and e'

The first segment s_1 connects a with some edge τ_1 of $\mathscr{A}(E_0)$ (note that $s_1 \neq e$ when e is shielded). Since s_1 is shielded, there exists another graph edge $e^+ \in E^*$ with a subsegment $s_1^+ \in \Xi(e^+)$ that connects a to τ_1. Clearly, we can choose e^+ with these properties to be the graph edge emanating from a nearest to e (either clockwise or counterclockwise).

We now prove, by induction on j, that e^+ satisfies condition (2) of the claim for all $s_1, \ldots, s_j$. This is clearly true for $j=1$. Next, assume that $j>1$ and that the assertion is true for $j-1$. Suppose that s_j connects two segments τ_{j-1} and τ_j of $\mathscr{A}(E_0)$ such that $u=\tau_{j-1} \cap e$ is the common endpoint of s_{j-1} and s_j, and $v=\tau_j \cap e$ is the other endpoint of s_j. (If $j=k$, then we take τ_j and v to be the other endpoint b of e.) If the face f of $\mathscr{A}(E_0)$ containing s_j is a crossing quadrilateral, then e and e^+ must cross the same pair of opposite edges of f, as no three edges of G are pairwise crossing, completing the induction step. Otherwise, since s_j is shielded, there is a graph edge $e' \in E^*$ and a subsegment $s' \in \Xi(e')$ that connects τ_{j-1} and τ_j on the same side of s_j as e^+. Three cases can arise:

- $e'=e^+$: The induction step is complete.
- *e' crosses τ_{j-1} at a point that lies between u and the crossing with e^+* (see Figure 3(i)): Since G is quasi-planar, e' cannot cross e or e^+. Moreover, e' cannot have an endpoint within the interior of the triangle Δ bounded by e, e^+, and τ_{j-1}, by the induction hypothesis and the fact that all faces of $\mathscr{A}(E_0)$ are simply connected. Hence, e' must end at a and lie inside Δ near a. However, this contradicts the choice of e^+ as the closest neighbor of e near a. Thus, this case is impossible.
- *e^+ crosses τ_{j-1} at a point that lies between u and the crossing with e'* (see Figure 3(ii)): In this case, e^+ cannot cross s_j or s' or terminate inside f.

Thus, it must meet τ_j. This completes the induction step and hence the proof of the claim.

Note that the same analysis also applies when $j=k$, that is, when τ_j is the endpoint b of e. Therefore, e and e^+ have the same pair of endpoints, contradicting the assumption that G has no parallel edges. This completes the proof of the lemma. ∎

As there are no shielded edges, the total number of edges of E^*, and thus also of E, is $O(n)$. This completes the proof of Theorem 1.1.

As noted at the beginning of the analysis, the same proof also applies to the case where the edges of G are drawn as Jordan arcs, except that one has to handle more carefully several terms used in the proof, such as the side of an edge, order of edges around a common vertex, etc.

3. Discussion

In this section we discuss some consequences of the above results.

Theorem 3.1. *Let $G(V,E)$ be a graph with n vertices that can be drawn in the plane with no four pairwise crossing edges. Then the number of edges of G is $O(n\log^2 n)$.*

Proof. We first estimate the number C of crossings between the edges of G. Let e be an edge of G, and let G_e be the subgraph of G consisting of all edges that cross e. Then G_e is a quasi-planar graph. Thus, by Theorem 1.1, the number of edges of G_e is $O(n)$, which implies that $C=O(n|E|)$. One can then combine this estimate with the analysis in [7], to conclude that $|E|=O(n\log^2 n)$. ∎

Corollary 3.2. *Let $k\geq 4$ be an integer, and let G be a graph with n vertices that can be drawn in the plane with no k pairwise crossing edges. Then the number of edges of G is $O(n\log^{2k-6} n)$.*

Proof. This is an immediate consequence of the analysis in [7], which proceeds by induction on k, based on the improved bound of Theorem 3.1 for $k=4$. ∎

Theorem 3.1 and Corollary 3.2 improve the bounds given in [7] by a factor of $\Theta(\log^2 n)$.

Several interesting problems are left open in this paper. The first problem is to find the best constant of proportionality in the bound of Theorem 1.1. A trivial lower bound is roughly $6n$, obtained by overlaying two edge-disjoint triangulations of the same point set. The constant 6 can be slightly increased.

Another open problem is as follows. For a quasi-planar graph G, let $\chi=\chi(G)$ be the smallest number with the property that the edges of G can be colored with χ colors, so that the edges in each color class do not cross each other (and thus form a planar graph). Clearly, if G has n vertices, then the number of edges of G is at most $3\chi(G)n$. Thus, a plausible conjecture is that $\chi(G)$ is bounded from above by a constant. Recall that this conjecture is true, with $\chi(G) \leq 5$, if there exists a plane drawing of G in which no three edges are pairwise crossing and the vertices are in convex position (see also [3, 4] for a weaker constant bound and for related results concerning more general classes of graphs). Moreover, if there exists such a drawing of G in which the vertices lie on two parallel lines, then one can easily show that $\chi(G) \leq 2$. Does there exist a constant upper bound for $\chi(G)$ when all edges of G cross a common line? A weaker conjecture is that there exists a subset E' of pairwise noncrossing edges of G such that $|E'| \geq \beta|E|$ for some absolute constant $\beta > 0$. The existence of such a subset E' would imply, by planarity, that $|E'| = O(n)$, which would provide another proof of Theorem 1.1.

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References

[1] V. Capoyleas and J. Pach: A Turán-type theorem on chords of a convex polygon, *J. Combin. Theory, Ser. B*, **56** (1992), 9–15.

[2] M. Golumbic: *Algorithmic Graph Theory*, Academic Press, New York, 1980.

[3] A. Gyárfás: On the chromatic number of multiple interval graphs and overlap graphs, *Discrete Math.*, **55** (1985), 161–166.

[4] A. Gyárfás and J. Lehel: Covering and coloring problems for relatives of intervals, *Discrete Math.*, **55** (1985), 167–180.

[5] A. V. Kostochka: On upper bounds for the chromatic number of graphs, in: *Modeli i metody optimizacii, Tom 10*, Akademiya Nauk SSSR, Sibirskoe Otdelenie, 1988, 204–226.

[6] J. Pach and P. K. Agarwal: *Combinatorial Geometry*, Wiley–Interscience, New York, 1995.

[7] J. Pach, F. Sharokhi, and M. Szegedy: Applications of the crossing number, *Proc. 10th Annual ACM Symp. on Computational Geometry* (1994), pp. 198–202. Also to appear in *Algorithmica*.

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